III

QUASI-ANALYTIC FUNCTIONS OF A REAL VARIABLE

1. *Borel's investigations and the concept of quasi-analytic function*

Analytic functions of a complex variable possess the property of being determined in their whole region of existence as soon as they are known in a region which is arbitrarily small. In fact, they are determined by their values and the values of all their successive derivatives at a single point z_0. This comes from the circumstance that they are developable in a Taylor's series in the neighborhood of the point, and the Taylor's development identifies, as we know, the analytic functions.

It was thought for a long time that the analytic functions were the only ones which were determined by their values and the values of their derivatives at a single point. It was Borel who first conceived and proved the existence of functions belonging to more general classes than that of analytic functions, which nevertheless possessed this same property; and he gave them the name *quasi-analytic*, by which they are now known.[1]

The functions which Borel considered are the *monogenic non-analytic* functions, defined in the complex plane, and on them he published a fundamental study in the collection

[1] *Les séries de fonctions analytiques et les fonctions quasi-analytiques*, Comptes Rendus, t. 154 (1912), p. 1491.

which bears his name.[1] But we shall not now take up the investigations of Borel, since for the present this study will not help us; and we shall only speak of the quasi-analytic functions considered in the real domain.

2. *Quasi-analytic functions of a real variable according to Denjoy's point of view*

This new method of treating quasi-analytic functions, which is much simpler than that of Borel, and which, as a matter of fact, is independent of that geometer's work, is due to Denjoy. Denjoy gave the exposition of his point of view and his results in a very remarkable article in the *Comptes Rendus*.[2] He considered quasi-analytic functions in a real interval, and showed that they could be defined by properties analogous to those which define analytic functions in such an interval, but which are merely *less restrictive*.

We shall first show that the functions which are analytic in a real interval, for instance, in the segment (0, 1), may be characterized by a condition of limitation imposed on their successive derivatives.

Let us suppose then that $f(z)$ is analytic on the segment $0 \leq x \leq 1$. This segment can be enclosed in a region D where $f(z)$ is still holomorphic and where its modulus does not exceed M. Let R be the shortest distance from the segment to the frontier of the region D. With any point X_0 of (0, 1) as center we can describe a circle C of radius R, interior to D. We apply accordingly the well-known formulae of Cauchy

[1] *Leçons sur les fonctions monogènes uniformes;* Paris, Gauthier-Villars (1917), edited by G. Julia. (See also Borel's lectures in The Book of the Opening of the Rice Institute (1912), vol. 2, pp. 399–430 [transl.]).

[2] Comptes Rendus, t. 173 (1921), p. 1329.

$$f(x_0) = \frac{1}{2\pi i}\int_c \frac{f(z)dz}{z - z_0},\ f^{(p)}(x_0) = \frac{p!}{2\pi i}\int_c \frac{f(z)dz}{(z - z_0)^{p+1}}$$

from which follow the formulae of limitation

$$|f| < M,\ |f^{(p)}| < \frac{p!M}{R^p}.$$

which are also well known.

In the last formula we may replace the $p!$ by its approximate value, given by Stirling's formula

$$\sqrt{2\pi p}\left(\frac{p}{e}\right)^p.$$

If now we notice that the expression

$$\frac{\sqrt[2p]{2\pi p M^2}}{e^R}$$

remains for all p less than a definite constant k, independent of p, we conclude, from the last inequality, the following one:

$$(C_0) \qquad \sqrt[p]{|f^{(p)}|} < kp.$$

Hence any function analytic on the segment (0, 1) satisfies the condition (C_0), for all p, with a proper value of the constant k. Conversely, a function which satisfies the condition (C_0) is analytic on the segment (0, 1), since this condition implies the convergence of the Taylor development about every point X_0 of that segment.

The condition (C_0), therefore, characterizes in the real domain the class of functions analytic on the segment (0, 1), and accordingly the functions of that class are completely determined by their values and the values of their successive derivatives at a point, let us say 0, of the segment.

Denjoy asked himself if it was not possible to enlarge the class (C_0) without making this property disappear. For this purpose he inserted factors which become infinitely great with p into the second member of the condition (C_0) and in this way defined classes (C_1), (C_2), ... more and more extensive, and characterized by conditions

$$(C_1) \qquad \sqrt[p]{|f^{(p)}|} < kp \log p$$

$$(C_2) \qquad \sqrt[p]{|f^{(p)}|} < kp \log p \log \log p,$$

and so forth. He verified that the functions of these different classes were still completely determined by their initial values, and preserved for them the name *quasi-analytic* functions, proposed by Borel.

3. *The Denjoy-Carleman theorem*

These results were proved rigorously by Denjoy himself in the article mentioned, but he was led by induction to announce a further proposition, without however being able to prove it completely. He noticed that the reciprocals of the second members of the inequalities given above are themselves the general terms of divergent series, less and less rapidly divergent, and forming, in fact, a classic scale of such series. He was thus led to announce the following theorem:

Let $f(x)$ be a function which is indefinitely differentiable in the interval $(0, 1)$ *and not analytic, and denote by M_0, M_1, M_2, ..., M_n, ... the absolute maxima of f, f', ..., $f^{(n)}$, ... in that interval; the function will be quasi-analytic, that is, completely determined in the whole interval by its value and the values of its derivatives in one point, if the series of positive terms*

$$\frac{1}{M_1} + \frac{1}{\sqrt{M_2}} + \dots + \frac{1}{\sqrt[n]{M_n}} + \dots$$

is divergent.

This theorem was completely proved a fortnight later by Carleman,[1] and on that account it is called the *Theorem of Denjoy-Carleman*. I gave a complete proof of it in the lectures which it was my privilege to give in Paris in 1923.[2] But the complete demonstration of this theorem encounters many little difficulties of detail, which have only a secondary interest. Hence I am going to replace Denjoy's statement of the theorem by another which is simpler and avoids these secondary complications, and which brings out the general idea of the proof in its fine simplicity.

We shall then for the moment be satisfied with proving the following theorem, which defines a very general class of quasi-analytic functions.

4. *Theorem*

The class of functions which are indefinitely differentiable in the interval (0, 1) *and whose successive derivatives satisfy the condition of limitation*

$$(C) \qquad \sqrt[p]{|f^{(p)}|} < k\phi(p)$$

where k is a constant and $\phi(p)$ a function which increases constantly and without limit, is a class of quasi-analytic functions if the series $\sum \frac{1}{\phi(p)}$, or (what amounts to the same thing) if the integral with infinite limit

$$\int_0^\infty \frac{d\phi}{\phi(p)}$$

is divergent.

[1] Comptes Rendus, t. 174 (1922), p. 373. Another demonstration, of entirely different character, was given by Carleman, in a session of the Fifth Congress of Scandinavian Mathematicians, Helsingfors, 1923.

[2] *Four lectures on analytic functions of a real variable.* Bulletin de la Société Mathématique de France, No. Jubilaire, 1924.

A function of the class (C) is then completely determined in the whole interval $(0, 1)$ by its value and the values of its successive derivatives at the point 0.

We can introduce several simplifications in the proof of the preceding theorem.

If two functions f_1 and f_2 take on the same initial values, as well as their derivatives, the initial values for their difference will vanish. In order that f_1 and f_2 be identical it is sufficient that their difference be identically zero. Now if the two functions f_1 and f_2 are of the class (C), their difference is of the same class. In order to prove the above theorem it will accordingly be sufficient to show that if a function of class (C) vanishes with all its derivatives at the point 0, the function vanishes identically in the interval $(0, 1)$.

It suffices also to prove the theorem for a not negative function f, since we can replace f by f^2, which also belongs to the same class (C).

Finally, we may assume that $f(x)$ satisfies the symmetry condition

$$f(x) = f(1 - x),$$

since we can, if need be, replace $f(x)$ by the function

$$F(x) = f\,[4\,x(1 - x)]$$

which satisfies this condition and also belongs to the class (C). This fact is easily verified by calculating the n^{th} derivative.

It is therefore sufficient now to prove the following theorem:

5. *Theorem*

A function $f(x)$ of class (C), which is not negative and which vanishes with all its derivatives at both ends of the interval $(0, 1)$, vanishes throughout the whole of that interval.

In order to prove the theorem we form, with Denjoy, the integral

$$F(z) = \int_0^1 f(x)e^{-zx}dx$$

considering z as a parameter. This integral defines an entire function $F(z)$ of the complex variable z, on the whole plane. This fact is immediate. Our theorem will then follow if we can prove that $F(z)$ is identically zero, for we shall have, for $z = 0$,

$$F(0) = \int_0^1 f(x)dx = 0,$$

and the function $f(x)$, being not negative, will be zero.

We see at once that the function $F(z)$ is bounded (since the same is true of e^{-zx}) to the right of the imaginary axis. If $F(z)$ were also bounded to the left of this axis, we could apply Liouville's theorem, and deduce that $F(z)$ is a constant, and therefore zero, since we know that $F(z)$ vanishes for z positively infinite. But we do not know how to make this proof. Denjoy evaded this difficulty by using ideas which testify to his great ingenuity, but succeed only with special forms of the function $\phi(p)$, introduced in the condition (C).

It was Carleman who found the real basis for the proof, and made known, at the same time, a very remarkable theorem of the theory of analytic functions. To this theorem I devoted the first of the lectures which I gave at Paris last year, and I refer you to that lecture for the proof.

Carleman's theorem is somewhat analogous to the theorem of Liouville, but whereas Liouville's theorem refers to a function considered over the whole plane, that of Carleman concerns the half plane. It says that a function $F(z)$, which is bounded in the half plane to the right of the imaginary axis and approaches 0 as z becomes infinite along the imaginary axis in either direction, will necessarily reduce to zero if the rapidity of decrease of the function exceeds a certain limit.

We can easily state such a limit.

Let $z = re^{\theta i}$. According to our hypothesis that F approaches zero, we can write

$$|F(re^{\pm\frac{\pi i}{2}})| < Ae^{-\psi(r)},$$

with $\psi(r)$ an increasing function of r which becomes infinite with r, and the more rapidly according as F tends more rapidly to zero.

This is now Carleman's theorem:

If $\psi(r)$ increases sufficiently fast so that the integral to infinity

$$\int^{\infty} \frac{\psi(r)dr}{r^2}$$

is divergent, we shall have identically $F(z) = 0$.

We are thus led, in order to apply this theorem to Denjoy's integral $F(z)$, to investigate how $F(z)$ decreases as z tends to infinity along the imaginary axis. For this purpose we make p consecutive integrations by parts, and observe that since $f(z)$ and its derivatives vanish at the two ends of the interval all the terms at these extremities are zero. Hence

$$F(z) = \int_0^1 \frac{f^{(p)}(x)}{z^p} e^{-zx}\, dx.$$

From this we have, by condition (C),

$$|F(re^{\pm\frac{\pi i}{2}})| < \left[\frac{k\phi(p)}{r}\right]^p.$$

Let us fix the integer p by the conditions

$$\phi(p) \leq \frac{r}{ke} < \phi(p+1)$$

whence, if we designate by ψ the function inverse to ϕ,

$$p \leq \psi\left(\frac{r}{ke}\right) < p+1.$$

We shall have *a fortiori*

$$|F(re^{\pm\frac{\pi i}{2}})| < Ce^{-\psi\left(\frac{r}{ke}\right)}.$$

In this way the condition of limitation on the function F is obtained by means of the inverse of the function ϕ.

In order that F be identically zero, it is therefore sufficient, in accordance with the above theorem from the theory of analytic functions, that the integral with infinite limit

$$\int^{\infty} \frac{\psi\left(\frac{r}{ke}\right)}{r^2}\,dr \text{ or (what amounts to the same thing)} \int^{\infty} \frac{\psi(r)}{r^2}\,dr$$

be divergent.

But this fact is precisely the consequence of condition (C), because the integral of condition (C) is reduced to the above integral by an integration by parts; and both integrals are divergent at the same time. We have, in fact, if n is the inverse of ϕ,

$$\int \frac{dn}{\phi(n)} = \frac{n}{\phi(n)} + \int \frac{n\,d\phi}{\phi^2} = \frac{n}{\phi(n)} + \int \frac{\psi(\phi)}{\phi^2}\,d\phi.$$

The function $f(x)$, by hupothesis, fails to be analytic; hence $\phi(n)$ surely exceeds n for certain properly chosen values of n, sufficiently great, and the last integral (in which ψ is positive) is infinite whenever the first is. It is clear, conversely, that under the same condition, the divergence of the last integral implies the divergence of the first integral, in the preceding relations.

The theorem is thus fully established.

6. *Identification of quasi-analytic functions by the order of decrease of the coefficients of their trigonometric development*

The investigations of Denjoy and Carleman are exceedingly interesting, but they leave a gap in the theory. In fact, they characterize the quasi-analytic functions by properties which cannot immediately be tested in terms of their analytic expression, and moreover do not lead to any practical method of analytic representation for those functions. This is true to such an extent, that one would be at a loss to define explicitly particular quasi-analytic functions, and one might even place their existence in doubt. I have shown, by applying to these special functions the methods which I utilized in the study of the approximation of functions, that the difficulties which I have just mentioned disappear if we consider the development of quasi-analytic functions in trigonometric series.

It is of course understood that this trigonometric representation is applicable only in the case of periodic functions. We shall consider therefore only a function $f(x)$ with period 2π. This restriction does not interfere with the generality of the theory, since it is always satisfied if we replace the variable by $\cos x$. This substitution, performed on a function of class (C) does not alter, in general, the condition of limitation for the derivatives appropriate to this class, but we shall not stop to prove the statement. There is nothing, in fact, to prevent us from studying the properties of the function when the substitution has been made.

In order to simplify the writing we shall consider an even function. The above mentioned substitution always leads to such a function. But we notice also that any function is the sum of an even and an odd function, and that the derivative of an odd function is an even function. Hence in any case, matters come back to the even function.

Let $f(x)$ then be an even function, of period 2π and indefinitely differentiable. It is expressible in a trigonometric series, indefinitely differentiable, of the form

$$f(x) = \frac{1}{2} a_0 + \sum_{1}^{\infty} a_k \cos kx.$$

We want to show that the various classes of quasi-analytic functions can be characterized by the law of limitation imposed on their Fourier coefficients a_k as well as by that imposed on their successive derivatives. This is the consequence of the formulae which I established in the preceding lecture.

We have seen that the condition

$$\sqrt[n]{|f^{(n)}|} < \phi(n)$$

implies as a consequence the condition

$$|a_k| < Ae^{-\psi\left(\frac{k}{e}\right)} \qquad (A \text{ const.}),$$

where the function ψ is the inverse of the function ϕ. Also, we saw that conversely the condition

$$|a_k| < Ae^{-\psi(k)}$$

implies as a consequence the condition

$$\sqrt[n]{|f^{(n)}|} < B\phi\left(\frac{2n}{\alpha}\right), \qquad (B \text{ const.}),$$

with the proviso that we can choose α, positive, so that the function $\psi(x)/x^{\alpha}$ is continually increasing. Finally, we know that in the case of a function $f(x)$ which is not analytic, the integrals with infinite limit

$$\int^{\infty} \frac{dn}{\phi(n)} \qquad \int^{\infty} \frac{\psi(k)\, dk}{k^2}$$

are both divergent or both convergent.

From this we have the following conclusion:

The class (C) of quasi-analytic functions which we have already defined may equally well be characterized by the condition of limitation

$$(C) \qquad |a_k| < Ae^{-\psi(k)} \qquad (k = 0, 1, 2, \ldots)$$

imposed on its Fourier coefficients, subject to the hypotheses: (1) that there exist a positive α such that the function $\psi(k)/k^{\alpha}$ shall be non-decreasing, and (2) that the integral

$$\int^{\infty} \frac{\psi(k)\, dk}{k^2}$$

shall be divergent.

Of course, these conditions are not assumed to hold in the interval (0, 1) as before, but in the interval of a complete period (of amplitude 2π), and, therefore, for all values of x (on account of the periodicity).

We can apply the above principle to the definition of the classes (C_1), (C_2) ..., which were considered originally by Denjoy, and which enter as particular cases in the above class (C), by giving to ψ the particular evaluations:

$$(C_1) \qquad \psi(k) = \frac{k}{\log k}$$

$$(C_2) \qquad \psi(k) = \frac{k}{\log k \, \log \log k}$$

and so on.

It is now possible to write down at once a quasi-analytic function which belongs to one of these classes. Thus, for instance, the function

$$f(x) = \Sigma\, e^{\frac{k}{\log k}} \cos kx$$

is a quasi-analytic function which belongs to the first Denjoy class (C_1).

By utilizing the results already obtained, it would not be difficult to characterize certain classes of quasi-analytic functions by the order of approximation which they admit by means of a finite expression of order n. This is not the method of definition which seems most natural to us, but we may notice here that Bernstein has taken this direction of approach and obtained quite recently some very interesting results.[1]

7. *Calculation of a quasi-analytic function for which the initial values are given*[2]

We take up the problem of determining a periodic quasi-analytic function, assumed to exist, by its initial value and the initial values of its derivatives for $x = 0$. It suffices evidently to be able to solve the problem for an even function, for as we have said, every function is the sum of an even and an odd function. The derivatives of odd order of the former vanish at the origin, and the same applies to the derivatives of even order of the latter. The determination of the two functions can thus be done independently. Finally, the determination of an odd function reduces to the calculation of its derivative, which is an even function, — a fact which justifies our statement.

The problem then is to determine the even quasi-analytic function

$$f(x) = \Sigma a_r \cos rx$$

on the basis of the following circumstances:

(1) We assume that the function exists and belongs to the class defined by the condition

$$|a_r| < Ae^{-\psi(r)} \tag{1}$$

[1] *Sur les fonctions quasi-analytiques de M. Carleman*, Comptes Rendus, t. 179 (1924), p. 743.

[2] The solution of this problem was given by Carleman, by developing the function in a series of polynomials [Comptes Rendus, t. 176 (1923), p. 64]. The solutions which we give here are much simpler.

where A is a given constant and $\psi(r)$ a given function which becomes infinite with r in such a way as to insure the infinite differentiability and the quasi-analyticity of the function.

(2) We assume that the function and its derivatives of even order take on the initial values $C_0, C_2, \ldots, C_{2i}, \ldots$

In accordance with this, we have the system of equations

$$(2) \qquad C_{2i} = \sum_{r=0}^{\infty} (-1)^i r^{2i} a_r \qquad (r = 0, 1, 2, \ldots),$$

to determine the Fourier coefficients.

We shall first show that the solution of the problem reduces to the successive solution of linear inequalities. The calculation of $f(x)$ depends, in fact, on a passage to the limit which rests on the indefinite repetition of the operation which we are going to describe once for all.

We assign two positive numbers ϵ and n, the first as small, the second, which is an integer, as large as we please. Consider the first n equations (2); without knowing the a_r, but making use of the conditions (1) which determine a definite order of convergence of the series, we can take in each sum of the second member of the first n equations (2), a sufficient number of terms so that the remainders of these n series will be $< \epsilon$ in absolute value. Suppose that for this purpose, to make our notions precise, we take the same number, $p+1$, of terms in each equation, although this is not necessary. We shall have the n inequalities:

$$(3) \quad \left| C_{2i} - \sum_{r=0}^{p} (-1)^i r^{2i} a_r \right| < \epsilon \qquad [i = 0, 1, 2, \ldots, (n-1)].$$

The function $f(x)$ is assumed to exist, so that its Fourier coefficients $a_o, \ldots, a_p$ satisfy the $(p+1)$ above mentioned inequalities (1) and the n inequalities (3). Hence if we consider the coefficients a_r as unknown these inequalities form a system of $n + p + 1$ linear inequalities among the

$a_0, a_1, \ldots, a_p$ which we are sure have solutions. The determination of a particular system of solutions is a problem which may be long, but the solution of it is elementary. Let us find then, by any method we like, a particular system of solutions $a'_0, a'_1, \ldots, a'_p$. We form thus the function

(4) $\phi(x) = a'_0 + a'_1 \cos x + a'_2 \cos 2x + \ldots + a'_p \cos px$

which may be considered as an approximate solution of the problem.

This is the method of calculation which must be repeated an infinite number of times, and leads to the solution of the problem.

We take an infinite sequence of pairs of values of ϵ, p. Let $(\epsilon_1, p_1), (\epsilon_2, p_2), \ldots (\epsilon_n, p_n), \ldots$ be these various pairs, ϵ_n approaching zero and p_n becoming infinite. To these successive pairs correspond successive functions $\phi_1, \phi_2, \ldots \phi_n, \ldots$ of the form (4). I say now that this sequence of functions ϕ_n is convergent, and that ϕ_n converges to the desired quasi-analytic function $f(x)$, as n becomes infinite.

In fact, all the functions ϕ_n are of the form (4) where the coefficients $a'_0, a'_1, a'_2, \ldots$ are subject to the conditions (1) and are consequently enclosed in intervals which insure the absolute and uniform convergence of the series and the quasi-analyticity of the sum function. The sets formed by the various values of the coefficients $a'_0, a'_1, \ldots$ respectively are bounded and consequently have at least one limit point. We can take from the sequence of the ϕ_n a first sequence where a'_0 converges, from this first a second sequence where a'_1 converges, and so on. In this way we determine in the limit a quasi-analytic function which admits the initial values C_{2i}, and this function is nothing else than the $f(x)$ desired, since $f(x)$ is the only one which can take on these initial values.

But even the complete sequence $\phi_1, \phi_2, \ldots, \phi_n, \ldots$

converges to $f(x)$. For if this sequence failed to converge there would be at least two limit points for the set of values given to at least one of the coefficients a', say a'_0. In this case we could, following the above method, determine two quasi-analytic functions with the same initial values but different coefficients a'_0. Hence the functions would be different, which is impossible.

8. *Modification of the above method of solution*

The method of solution which I have just developed is the one which is probably closest to the nature of the problem and clarifies it most. But we may look for a more elegant procedure, and, in fact, imitate the one already invented by Carleman.[1] This method avoids the resolution of inequalities by means of an infinite process depending on the method of least squares. This is the way I followed in my lectures last year at Paris, but the method I used then may again be modified and put in a more satisfactory form, which I will now proceed to outline.

Take again two numbers ϵ and n, the first as small, and the second integral and as large as we please. In virtue of the assumed conditions, namely

$$(1) \qquad |a_r| < Ae^{-\psi(r)},$$

we can choose the integer p great enough so that the coefficients $a_0, a_1, \ldots, a_p$ of the quasi-analytic function desired will satisfy the n conditions:

$$(2) \qquad \left(C_{2i} - \sum_0^p r^{2i} a_r\right)^2 < \frac{\epsilon}{n}, \quad (i = 0, 1, \ldots n-1).$$

We write down the quadratic form in the $p+1$ variables $y_0, y_1, y_2, \ldots y_n$,

$$(3) \qquad F(y) = \sum_{i=0}^{n-1} \left(C_{2i} - \sum_0^p r^{2i} y_i\right)^2 + \epsilon \sum_0^p y_r^2 e^{\psi(r)}.$$

[1] Comptes Rendus, t. 176 (1923), loc. cit.

This is a definite positive form. Let us find its minimum. The values $a'_0, a'_1, \ldots, a'_p$ of the y which give a minimum are obtained by solving a system of linear equations.

This minimum of $F(y)$ is not greater than the value of $F(y)$ when we put for $y_0, y_1, y_2, \ldots$ the coefficients a_0, a_1, $a_2, \ldots$ of the desired function $f(x)$, hence less than the quantity

$$n\frac{\epsilon}{n} + \epsilon A^2 \sum_{r=0}^{p} e^{-\psi(r)} = (1 + A^2B)\epsilon$$

where we designate by B the value of the convergent series of exponentials.

It follows therefore that the values $a'_0, a'_1, \ldots a'_p$ which make $F(y)$ a minimum make each term of the sum (3) less than that limit. Hence we have the conditions

$$(4) \qquad \left(C_{2i} - \sum_{0}^{p} r^{2i} a'_r\right)^2 < (1 + A^2B)\epsilon$$

(5) $a'^2_r e^{\psi(r)} < (1 + A^2B)$, whence $a'_r < \sqrt{1 + A^2B}\; e^{-\frac{\psi(r)}{2}}$.

By means of the above process, to each system ϵ, p, we make correspond a function

$$\phi(x) = a'_0 + a'_1 \cos x + \ldots + a'_p \cos px$$

whose coefficients satisfy the conditions (4) and (5). It is an approximate solution in the same sense as in the previous method. As ϵ approaches 0 and p becomes infinite this approximate solution approaches the desired quasi-analytic function, since the conditions (5) insure the convergence of the process and the quasi-analyticity of the limit as in the previous method.

9. *On the existence of a quasi-analytic function which admits the initial values*

The success of the methods which we have just explained is based on the hypothesis that the quasi-analytic function $f(x)$ exists which takes on the given initial values. What happens if, given the initial values and the class of the quasi-analytic function, we are not sure of the existence of a solution?

Suppose we apply the first method, which consists in solving systems of inequalities. Two cases are possible.

One alternative is that at a certain stage we find ourselves stopped in the sequence of operations, by encountering a system of incompatible inequalities. In this case we know that the problem is impossible and that there is no quasi-analytic function of the proposed class which with its derivatives takes on the initial values.

The other alternative is that after having carried the calculation more or less far, we stop before having encountered an incompatibility. In this second case we shall have obtained a more or less approximate solution of the problem, but we will not be sure of the existence of an exact solution, since it is always possible that we may meet an incompatibility by pushing the calculation further.

The question of finding a criterion of consistence for the initial values of a quasi-analytic function, of a certain class, and its derivatives, is thus before us. But it seems exceedingly difficult to solve. We shall not undertake it here.

Nevertheless, in a Note which we append to this lecture we shall explain some results with regard to a related but much simpler problem, that of constructing a function, quasi-analytic or not, taking on with its successive derivatives a system of given initial values.